Likelihood Ratio Introduction

A new feature added to the Generic Monte Carlo Engine with Volatility Skew is the Likelihood Ratio method for calculating risk sensitivities. The Monte Carlo engine is used for pricing and risk management of various products

The Local Spline Dupire volatility calibration serves its intended purpose to produce a local volatility surface which can be used to calculate the price and risk sensitivities of various products, based on Monte Carlo simulations or the PDE approach. The Likelihood Ratio method serves its intended purpose to calculate risk sensitivities in the Monte Carlo approach.

The key model assumptions are:

a. The asset values can be accurately expressed using a volatility skew model.
b. The interest rates are deterministic.
c. The volatilities of the exchange rates are flat (constant).
d. The pairwise correlations between assets are constant.
e. The pairwise correlations between assets and exchange rates are constant.
f. The future dividend dates, amounts and yields are known and fixed. Dividend yields are treated to preserve the dividend cash-flows.

The main risk uncertainties are the implied volatility surfaces of the underlying assets. Other risk uncertainties are correlations among assets and between assets and exchange rates.

The Local Spline Dupire volatility calibration can be used to price various products based on either Monte Carlo simulations or the PDE approach. The current implementation of the Likelihood Ratio method can be used when the parameter IsDistribution is set to True and the payoff function depends on the asset prices only at maturity.

The accuracy of the price and risk sensitivities produced by the Monte Carlo engine depends on the number of simulations. The convergence levels for each specific product will be estimated separately.

The generic Monte Carlo engine with volatility skew generates random paths for one or more underlying assets. These paths are used to calculate the price and risk sensitivities of various contingent claims based on these underliers. The model incorporates the volatility skew by assuming that the volatility is given either by a function of time to maturity and spot (local volatility) or by an implied distribution at maturity, depending on the simulation method.

The generic Monte Carlo simulation produces a set of random paths which are assumed to be representative of the probability space defined by the full range of possible asset prices (see *https://finpricing.com/FinPricing-ProductBrochure.pdf*).

The Local Spline (LS) calibration assumes that, in the log-moneyness space, the local volatility dependency on time is piecewise-constant, while the dependency on strike is given by a Bezier curve. Thus, local volatility is defined as a set of curves, one for each maturity in the implied volatility table. The curves are defined by uniformly spaced control points between the lowest and highest available strikes.

Users can specify the number of control points and a penalty term, $P \geq 0$, which restricts the arc length of the curve in order to reduce oscillations and enforce smoothness. Each curve is the solution of a least-squares optimization problem that minimizes the differences between model call prices and market call prices. The model call prices are determined by solving a numerical PDE for either the Fokker-Planck equation for the probability density function or the Dupire equation for call prices.

Under the sticky strike rule, when the current spot differs from the volatility spot, the surface is recalibrated. This introduces numerical instability since the minimizing routine may reach a different solution.

The new method, named LS Dupire, avoids the recalibration step by deriving the local volatilities from the PDE call grid, using the Dupire formula. The Dupire equation in the log-moneyness space is ([5]):

$$\frac{\partial c}{\partial t} = \frac{1}{2}\sigma^2(k,t)\cdot\left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)$$

where

$\sigma = \sigma(k,t)$	the local volatility
$k = \log\left(\frac{K}{F_{0,t}}\right)$	the strike in the log-moneyness space
K	the strike
$F_{0,t}$	the forward price from time 0 to time *t*
$c = c(k,t) = \frac{C(K,t)}{F_{0,t}}$	the normalized call price
$C(K,t)$	the undiscounted Black-Scholes call price

For the new method, the volatility is calibrated to the volatility spot using the same objective function as the LS calibration. The new method can fit the volatilities using either Bezier curves or monotone cubic splines (the latter being preferred). The control points for the monotone cubic splines are the available market quotes (which are not uniformly spaced). If the current spot differs from the volatility spot, implied volatilities are derived from the call prices on the PDE mesh from the initial calibration. The local volatilities for the current spot are obtained from the implied volatilities using the Dupire equation:

$$\sigma^2(K,t) = \frac{2\frac{\partial \upsilon}{\partial t} + \frac{\upsilon}{t} + 2K\left[r(t) - q(t)\right]\frac{\partial \upsilon}{\partial K}}{K^2\left[\frac{\partial^2 \upsilon}{\partial K^2} - d\sqrt{t}\left(\frac{\partial \upsilon}{\partial K}\right)^2 + \frac{1}{\theta}\left(\frac{1}{K\sqrt{t}} + d\,\frac{\partial \upsilon}{\partial K}\right)^2\right]}$$

where

$\sigma = \sigma(K,t)$	the local volatility
$\upsilon = \upsilon(K,t)$	the implied volatility
K	the strike
$r(t)$	the interest rate
$q(t)$	the dividend yield
$d = \frac{\ln\left(\frac{S}{K}\right) + \int_0^t \left[r(u) - q(u)\right]du}{\upsilon\sqrt{t}} + \upsilon\sqrt{t}$	term coming from the Black Scholes formula
S	the current spot price

The new method is stable when the market changes because the spot affects only the *d* term. Moreover, the resulting local volatility surface is smooth because the implied volatilities are arbitrage-free.

The Monte Carlo engine calculates risk sensitivities using the finite difference (FD) approach. This method is simple to understand and implement but it has a few well-known drawbacks:

- It requires a large number of simulations to achieve reasonable convergence.
- Each risk sensitivity requires at least one additional valuation.

Thus the FD approach is slow and computationally demanding.

The Likelihood Ratio method (LRM) assumes that the price depends on the selected risk factor only through the probability distribution. More precisely, consider the value of an option on a basket of *n* assets:

$$V = \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n) f(S_1, S_2, \ldots, S_n) dS$$

where h is the payoff function which depends only on the spot prices at maturity, and f is the terminal joint distribution of the assets $S_1, S_2, \ldots, S_n$. Let θ be a variable that f depends on but h does not. Then, the price sensitivity with respect to θ is:

$$\begin{aligned}\frac{\partial V}{\partial \theta} &= \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n) \frac{\partial}{\partial \theta} f(S_1, S_2, \ldots, S_n) dS \\ &= \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n) \left(\frac{\partial}{\partial \theta} \log f(S_1, S_2, \ldots, S_n) \right) f(S_1, S_2, \ldots, S_n) dS\end{aligned}$$

Thus, the risk sensitivities are estimated along the original simulation paths using a payoff function adjusted by the likelihood ratio $\frac{\partial \log f}{\partial \theta}$. This makes LRM more efficient than FD and the resulting sensitivities more stable to the number of simulation paths used. The Appendix provides further details regarding the LRM for the Gaussian copula as well as comparison tests between LRM and FD.

It should be mentioned that the current implementation of LRM can be used only when the IsDistribution parameter is set to True and the payoff function depends on the spot prices only at maturity.

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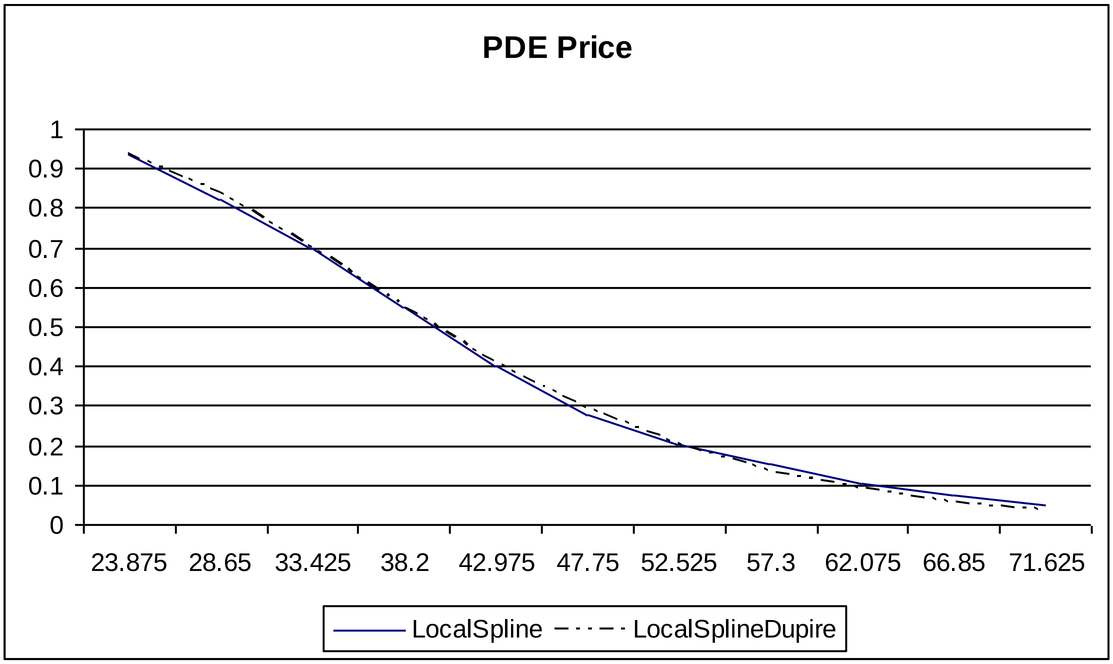
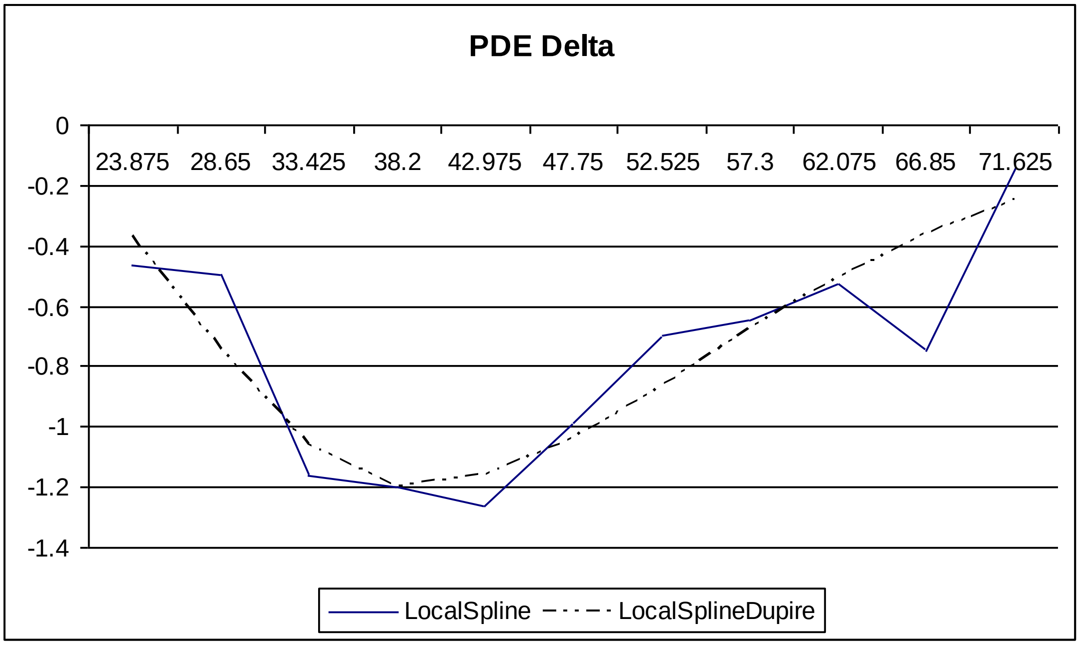
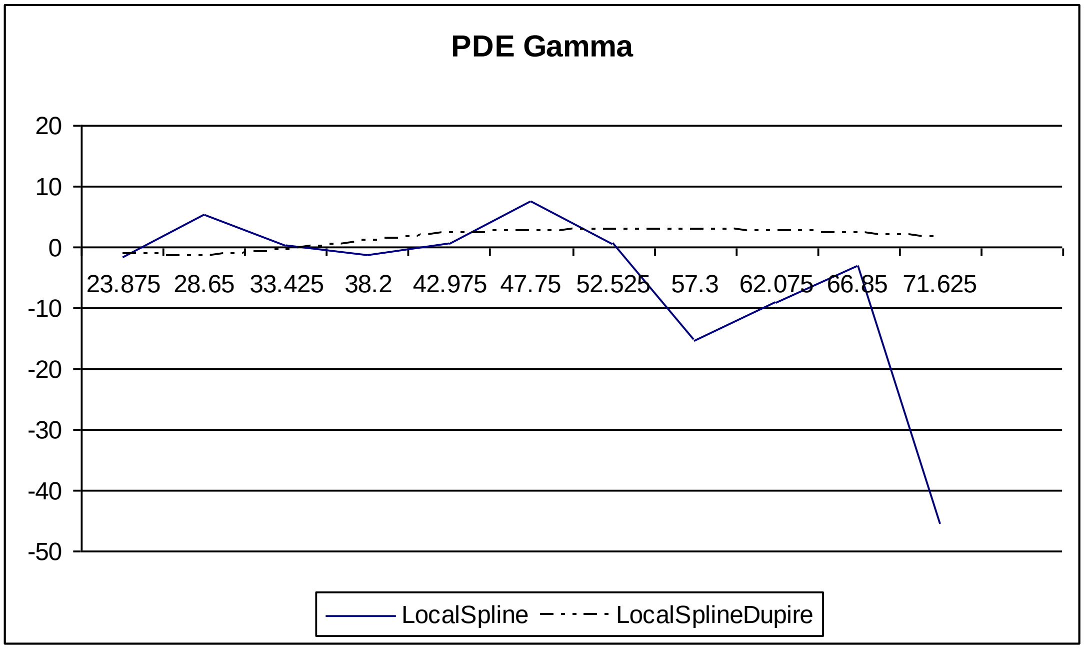
The graphs below show the changes in PDE price and risk sensitivities with respect to the spot value. The changes in price and discount rho are minimal, while changes in gamma and vega are substantial:

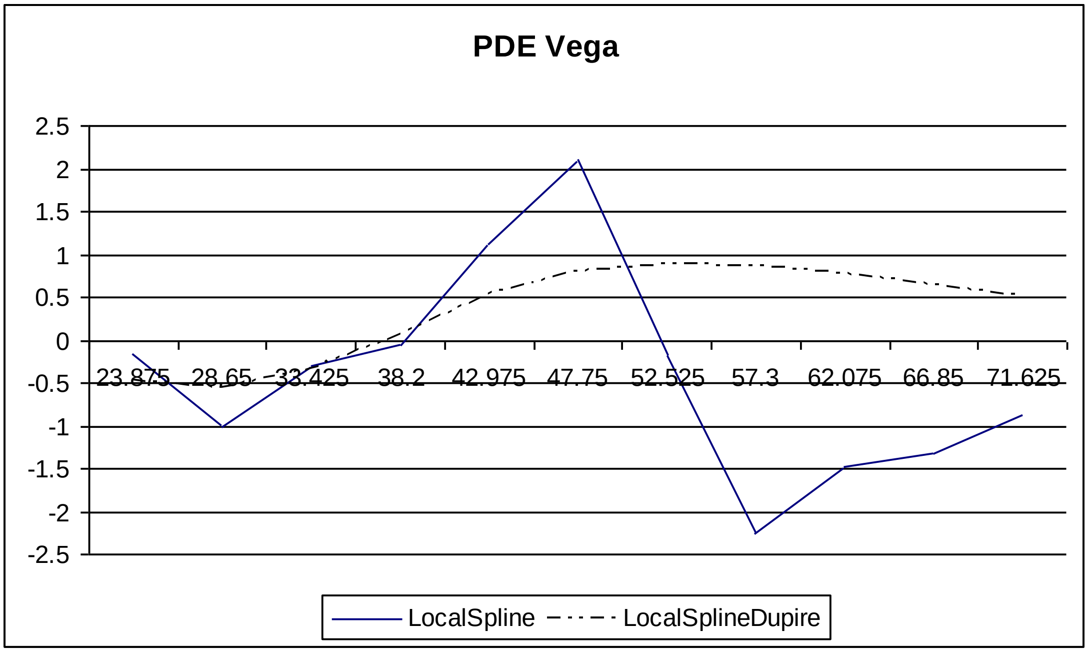


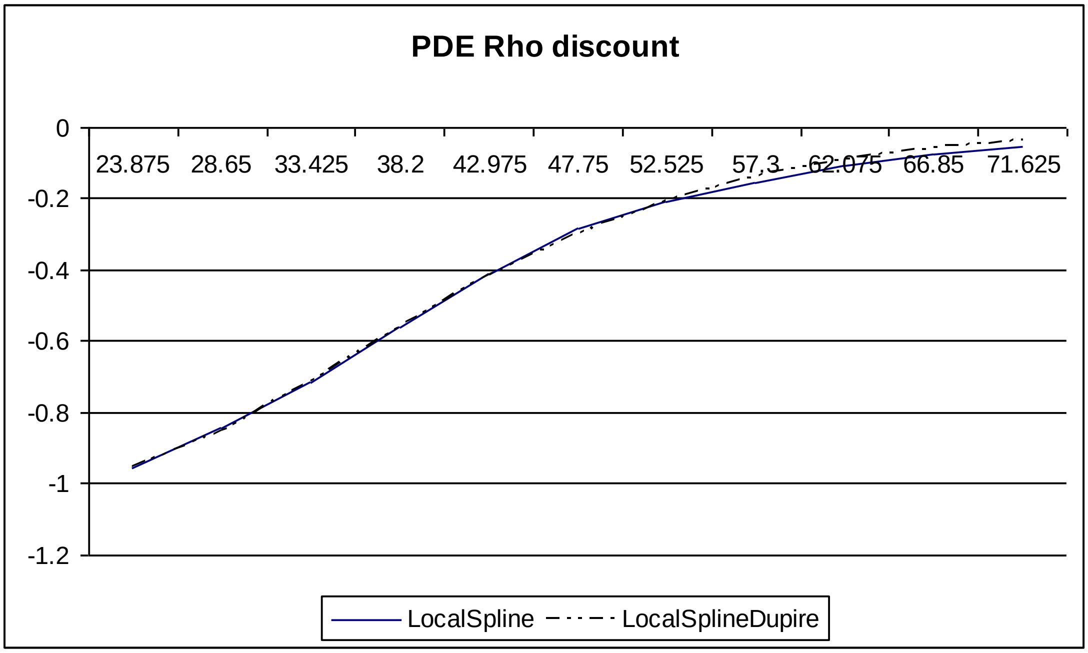


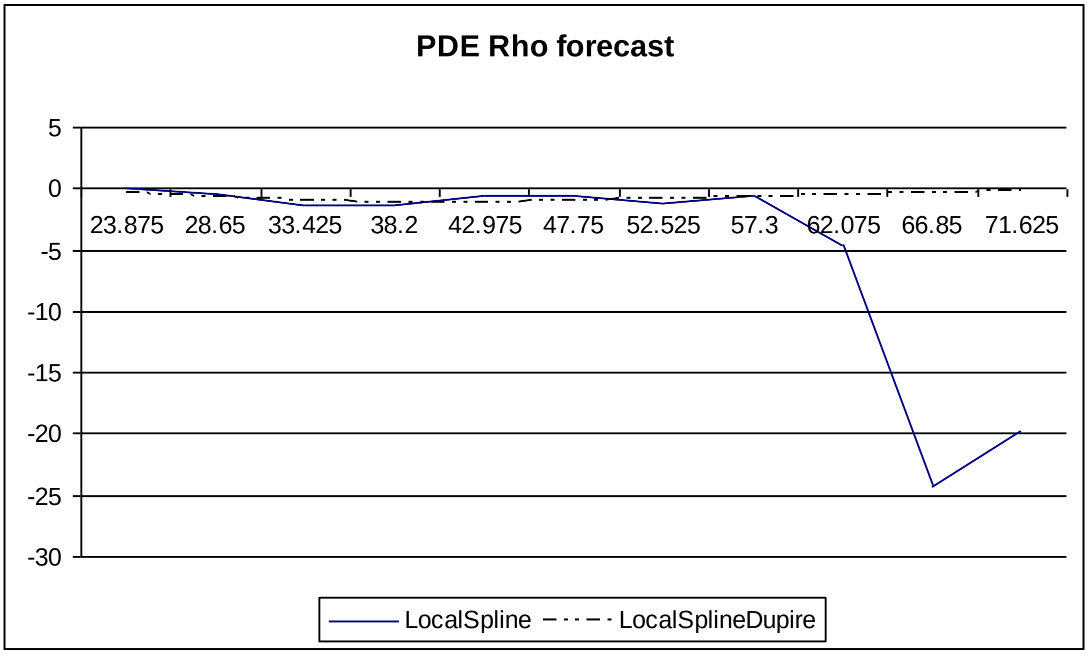
PDE Delta
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
23.875
28.65
33.425
38.2
42.975
47.75
52.525
57.3
62.075
66.85
71.625
LocalSpline
LocalSplineDupire



PDE Gamma
20
10
0
-10
-20
-30
-40
-50
23.875
28.65
33.425
38.2
42.975
47.75
52.525
57.3
62.075
66.85
71.625
LocalSpline
LocalSplineDupire


PDE Vega
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
23.875
28.65
33.425
38.2
42.975
47.75
52.525
57.3
62.075
66.85
71.625
LocalSpline
LocalSplineDupire


PDE Rho discount
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
23.875
28.65
33.425
38.2
42.975
47.75
52.525
57.3
62.075
66.85
71.625
LocalSpline
LocalSplineDupire



Recall that under LRM the risk sensitivities are obtained by adjusting the payoff function:

$$\frac{\partial V}{\partial \theta} = \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n)\left(\frac{\partial}{\partial \theta}\log f(S_1, S_2, \ldots, S_n)\right) f(S_1, S_2, \ldots, S_n) dS$$

Similarly, second order derivatives are given by:

$$\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n)\left(\frac{\partial^2}{\partial \theta_1 \partial \theta_2}\log f(S_1, S_2, \ldots, S_n)\right) f(S_1, S_2, \ldots, S_n) dS +$$

$$+ \int_{[0,\infty]^n} h(S_1, S_2, \ldots, S_n)\left(\frac{\partial}{\partial \theta_1}\log f(S_1, S_2, \ldots, S_n)\right)\left(\frac{\partial}{\partial \theta_2}\log f(S_1, S_2, \ldots, S_n)\right) f(S_1, S_2, \ldots, S_n) dS$$

The joint probability density function is:

$$f(S_1, S_2, \ldots, S_n) = c(F_1(S_1), F_2(S_2), \ldots, F_n(S_n))\prod_{j=1}^{n} f_j(S_j)$$

where $f_j\left(S_j\right)$ are the marginal densities, $F_j\left(S_j\right)$ the marginal cumulative distributions and $c\left(F_1\left(S_1\right), F_2\left(S_2\right), \ldots, F_n\left(S_n\right)\right)$ the copula term. The Gaussian copula is:

$$c\left(F_1\left(S_1\right), F_2\left(S_2\right), \ldots, F_n\left(S_n\right)\right) = \frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} U^T \left(\Sigma^{-1} - I\right) U \right)$$

where Σ is the correlation matrix and $u_i = N^{-1}\left(F_i\left(S_i\right)\right)$.

In this case, the likelihood ratio becomes:

$$\frac{\partial}{\partial \theta} \log f\left(S_1, S_2, \ldots, S_n\right) = -\frac{1}{2}\left(\frac{\partial}{\partial \theta}\left(U^T \left(\Sigma^{-1} - I\right) U\right)\right) + \frac{\partial}{\partial \theta} \sum_{j=1}^{n} \log f_j\left(S_j\right)$$

When the parameter IsDiagonalGamma is False, gamma is calculated as shown below. Let $f\left(S_i, \alpha\right) = f\left(S_1 + \alpha b_1, \ldots, S_i, \ldots, S_n + \alpha b_n\right)$, where $b_j, j \neq i$ represents the change in S_j with respect to a move in S_i. Also let:

$$f_{uu}^{i} = f\left(S_i(1+\varepsilon), 1\right)$$

$$f_{fu}^{i} = f\left(S_i, 1\right)$$

$$f_{fd}^{i} = f\left(S_i, -1\right)$$

$$f_{dd}^{i} = f\left(S_i(1-\varepsilon), -1\right).$$

The likelihood ratio for the gamma we want is $\frac{\partial^2}{\partial S_i \partial \alpha} \log f + \left(\frac{\partial}{\partial S_i} \log f\right)\left(\frac{\partial}{\partial \alpha} \log f\right)$ which is estimated as follows:

$$\frac{\partial^2}{\partial S_i \partial \alpha} \log f = \frac{\log\left(f_{uu}^{\,i}\right) - \log\left(f_{fu}^{\,i}\right) - \log\left(f_{fd}^{\,i}\right) + \log\left(f_{dd}^{\,i}\right)}{\left(\varepsilon S_i\right)^2}$$

$$\frac{\partial}{\partial S_i} \log f = \frac{\frac{1}{2}\left(\log\left(f_{uu}^{\,i}\right) - \log\left(f_{fu}^{\,i}\right) + \log\left(f_{fd}^{\,i}\right) - \log\left(f_{dd}^{\,i}\right)\right)}{\varepsilon S_i}$$

$$\frac{\partial}{\partial \alpha} \log f = \frac{\frac{1}{2}\left(\log\left(f_{uu}^{\,i}\right) + \log\left(f_{fu}^{\,i}\right) - \log\left(f_{fd}^{\,i}\right) - \log\left(f_{dd}^{\,i}\right)\right)}{\varepsilon S_i}$$